Quasi-projective covers of right $S$-acts

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Abstract. In this paper $S$ is a monoid with a left zero and $A_S$ (or $A$) is a unitary right $S$-act. It is shown that a monoid $S$ is right perfect (semiperfect) if and only if every (finitely generated) strongly flat right $S$-act is quasi-projective. Also it is shown that if every right $S$-act has a unique zero element, then the existence of a quasi-projective cover for each right act implies that every right act has a projective cover.

1 Introduction and Preliminaries

Let $S$ be a monoid. For right $S$-acts $A$ and $B$, $A$ is called $B$-projective or projective relative to $B$ if for every right $S$-act $C$, every homomorphism $f : A \to C$ can be lifted with respect to every epimorphism $g : B \to C$, that is there exists a homomorphism $h : A \to B$ such that $f = gh$. $A_S$ is called projective if it is projective relative to every right $S$-act. Also $A$ is called quasi-projective if $A$ is $A$-projective and is called weakly-projective if $A$ is projective relative to $S_S$ ([1, 7]). There are quite a few papers describing projective acts and their generalizations. Some other generalizations of projectivity are principal weak projectivity, Rees weak projectivity and principal Rees weak projectivity, see [6]. Quasi-projective acts have been studied by Ahsan and Saifullah [1]. Also the concept of weakly-projective acts have been introduced by Knauer and Olthmanns [7]. In this paper we study the concept of quasi-projective cover. Recall that over a monoid $S$, an $S$-act $A$ has a projective cover $P$ if there is an epimorphism $f : P \to A$.
such that, $P$ is projective and $f|_C : C \rightarrow A$ is not epimorphism for every subact $C$ of $P$ (see [5]). Similar to projective cover (as above) we can define quasi-projective cover, noting that $P$ has to be quasi-projective in this case. Monoids which have a projective cover for each right act are called right perfect monoids. For more details concerning covers of acts, see [2, 3, 4, 8]. In [2], Fountain proved that a monoid $S$ is right perfect if and only if every strongly flat right $S$-act is projective. From this point of view, we prove that for a monoid $S$ to be right perfect it is enough to show that every strongly flat right $S$-act is quasi-projective (see Theorem 2.5). Also we give a characterization for monoids for which every cyclic strongly flat act is projective. To give the main result, we focus our attention on right $S$-acts which have a unique zero. It is shown that if each right $S$-act has a quasi-projective cover, then $S$ is right perfect.

Modifying the proof of Lemma 1 of [1], we can deduce the following lemma.

**Lemma 1.1.** ([1]) Let $S$ be a monoid with a left zero and $\varphi : A_S \rightarrow B_S$ be an $S$-epimorphism. If $A_S \sqcup B_S$ is quasi-projective, then $B$ is a retract of $A$.

By the above lemma it is easy to see that over a monoid $S$ with a left zero an $S$-act (a finitely generated $S$-act) $A_S$ is projective if and only if there exists an epimorphism $g : P \rightarrow A$ such that $P$ is a (finitely generated) projective right $S$-act and $P_S \sqcup A_S$ is quasi-projective. This fact implies the following theorem:

**Theorem 1.2.** Suppose $S$ is a monoid with a left zero and $X$ is a property of acts which is preserved under coproduct and is weaker than projectivity (such as strongly flatness, flatness and etc.), then the following are equivalent:

(i) Every (finitely generated) right $S$-act with property $X$ is quasi-projective.

(ii) Every (finitely generated) right $S$-act with property $X$ is projective.

By Theorem 4.10.5 of [5] and Theorem 1.2, the following result holds.

**Corollary 1.3.** Over a monoid $S$ with a left zero the following are equivalent:

(i) Every principally weakly flat right $S$-act is quasi-projective.
(ii) Every weakly flat right $S$-act is quasi-projective.

(iii) Every flat right $S$-act is quasi-projective.

(iv) Every flat right $S$-act is projective.

(v) $S = \{1\}$

From Theorem 4.11.8 of [5] and Theorem 1.2, we can deduce the following Corollary.

Corollary 1.4. Suppose $S$ is a monoid with a left zero, then the following are equivalent:

(i) All finitely generated right $S$-acts which satisfy Condition (P) are quasi-projective.

(ii) Every right reversible submonoid of $S$ contains a left zero.

Recall that a right ideal $K$ of a monoid $S$ satisfies Condition (LU) if for every $x \in K, x \in Kx$ ([5]).

Proposition 1.5. Let $S$ be a commutative monoid, then the following are equivalent:

(i) All quasi-projective acts over $S$ are flat.

(ii) All quasi-projective acts over $S$ are weakly flat.

(iii) All quasi-projective acts over $S$ are principally weakly flat.

(iv) $S$ is a regular monoid.

Proof. (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii) are obvious.

(iii)$\Rightarrow$(iv). It is easy to see that over commutative monoids every cyclic act is quasi-projective. Thus for every $s \in S$, $\frac{S}{ss}$ is quasi-projective and so is principally weakly flat by assumption. Hence $sS$ satisfies Condition (LU) and so $s$ is regular.

(iv)$\Rightarrow$(i). It is well known that over a commutative regular monoid $S$, every act is flat. 

\square
2 Semiperfect and perfect monoids with a left zero

Recall that a monoid $S$ is right semiperfect if all cyclic strongly flat right $S$-acts are projective ([9]). In this section we give a new characterization of semiperfect and perfect monoids with a left zero. We present some results that we need in the sequel.

**Proposition 2.1.** Let $B_S$ be an $A_S$-projective $S$-act. If $C_S$ is either an $S$-homomorphic image or an $S$-subact of $A_S$, then $B_S$ is $C_S$-projective.

**Proof.** Clearly, if $C_S$ is a homomorphic image of $A_S$, then the result holds. Thus suppose $C_S$ is a subact of $A_S$ and consider an $S$-epimorphism $f : C \to \bar{C}$ and an $S$-homomorphism $g : B \to \bar{C}$ where $\bar{C}$ is a right $S$-act. Let $\rho = \ker f \cup \Delta_A$ where $\Delta_A$ is the diagonal relation on $A$. Clearly $\bar{C} \simeq C/\ker f$. Thus if $\pi : A \to A/\rho$ is the natural epimorphism, then $\pi$ is an extension of $f$. Since $B$ is $A$-projective, there exists $h : B \to A$ such that $\pi \circ h = g$. It is easy to see that $h(B) \subseteq C$ and so $h$ is an $S$-homomorphism from $B$ to $C$, which proves that $B$ is $C$-projective. \hfill \Box

One can easily see the following result.

**Lemma 2.2.** Suppose $S$ is a monoid and $A_S$ is a right $S$-act, then:

(i) If $A$ is a cyclic right $S$-act, then $A$ is projective if and only if $A$ is weakly-projective.

(ii) If $S$ contains a left zero and $A = \coprod_{i \in I} A_i$ is weakly-projective, then $A_i$ is weakly-projective for every $i \in I$.

**Lemma 2.3.** Suppose $S$ is a monoid with a left zero. If every finitely generated (strongly flat) right $S$-act has a quasi-projective cover, then every finitely generated (strongly flat) right $S$-act has a projective cover.

**Proof.** By Lemma 1.1 of [4] (Proposition 3.13.14 of [3] and Proposition 1.6 of [4]), it is sufficient to show that every cyclic (strongly flat) right $S$-act has a projective cover. Let $M = mS$ be a cyclic (strongly flat) right $S$-act and $\varphi : F \to M$ be an epimorphism such that $F$ is a free $S$-act. Note that $F$ can be regarded as a cyclic right $S$-act, because if $F = \coprod_{i \in I} a_i S$ and $m = \varphi(a_j t)$ for some $t \in S$ and $j \in I$, then $\varphi|_{a_j S} : a_j S \to mS$ is an
epimorphism. Thus if $F$ is not cyclic we can consider the new epimorphism replace $\varphi$. Clearly, $F_S \sqcup M_S$ is finitely generated (strongly flat) and has a quasi-projective cover $Q$ with an epimorphism $\psi : Q \to F_S \sqcup M_S$. Since $F$ is cyclic, $Q$ is finitely generated with two generators. If $F = aS$, then there exist $p, q \in Q$ such that $\psi(p) = m, \psi(q) = a$ and $Q = pS \sqcup qS$. Thus $\pi_F \circ \psi : Q \to F$ is an epimorphism. Since $F$ is projective, there exists a homomorphism $h : F \to Q$ such that $\pi_F \circ \psi \circ h = 1_F$ and hence $h$ is a coretraction. Since $F_S \simeq S_S$ and $h$ is a monomorphism, $S_S$ is a subact of $Q$. Thus by Proposition 2.1, $Q$ is weakly-projective and by Lemma 2.2(i), it is projective. Clearly $pS$ is the projective cover of $M$.

By the following theorem, we show that for a monoid $S$ with a left zero to be semiperfect it is enough to show that every finitely generated strongly flat right $S$-act has a quasi-projective cover.

Theorem 2.4. For a monoid $S$ with a left zero the following are equivalent:

(i) $S$ is right semiperfect.

(ii) Every finitely generated strongly flat right $S$-act has a quasi-projective cover.

(iii) Every finitely generated strongly flat right $S$-act has a weakly-projective cover.

(iv) Every cyclic strongly flat right $S$-act has a weakly-projective cover.

(v) Every left collapsible submonoid of $S$ contains a left zero (Condition (K)).

Proof. (i)$\Rightarrow$(ii), (i)$\Rightarrow$(iii). By Proposition 3.13.14 of [5] are clear. (iii)$\Rightarrow$(iv) is clear. (ii)$\Rightarrow$(i). By Lemma 2.3, every finitely generated strongly flat right $S$-act $A_S$, has a projective cover and so it is projective by Proposition 1.7 of [4]. (iv)$\Rightarrow$(i). If $A = aS$ is a strongly flat right $S$-act, then every cover of $A$ is cyclic. Now the result follows by Proposition 1.7 of [4] and Lemma 2.2(i). The equivalence of (i) and (v) follows by Theorem 4.11.2 of [5].

Recall that a monoid $S$ satisfies Condition(A) if every right $S$-act satisfies the ascending chain condition for its cyclic subacts ([5]). Fountain in [2], proved that a monoid $S$ is right perfect if and only if every strongly
flat right $S$-act is projective. The next theorem improves this result by the notion of quasi-projectivity.

**Theorem 2.5.** Let $S$ be a monoid with a left zero. The following are equivalent:

(i) $S$ is right perfect.

(ii) Every strongly flat right $S$-act is quasi-projective.

(iii) $S$ satisfies Condition (A) and every finitely generated strongly flat right $S$-act has a quasi-projective cover.

(iv) $S$ satisfies Condition (A) and every cyclic strongly flat right $S$-act has a weakly-projective cover.

**Proof.** (i)$\Rightarrow$(ii) is clear. (ii)$\Rightarrow$(i). Suppose every strongly flat right $S$-act is quasi-projective. Then by Theorem 1.2, every strongly flat right $S$-act is projective. Thus $S$ is right perfect by Theorem 1.8 of [4]. The equivalences of (i) and (iii), and also (i) and (iv) follow by Theorem 4.11.6 of [5] and Theorem 2.4.

Now we state the main result.

**Theorem 2.6.** Suppose $S$ is a monoid with a left zero and every right $S$-act has only one zero element. If every right $S$-act has a quasi-projective cover, then $S$ is right perfect.

**Proof.** We show that every right $S$-act has a projective cover. Suppose $M_S$ is a right $S$-act and $\phi : F \to M$ is an epimorphism such that $F_S$ is a free $S$-act. Let $F' = F - \{\theta_F\}$ and $M' = M - \{\theta_M\}$ and $B = F' \sqcup M' \sqcup \theta$, where $\theta$ is the one-element right $S$-act. Then $B$ is a right $S$-act by the right $S$-action, $\theta.s = \theta$ and

$$
    b.s = \begin{cases} 
        \theta, & \text{if } bs = \theta_F \text{ or } \theta_M; \\
        bs, & \text{otherwise} 
    \end{cases}
$$

for every $s \in S$ and $b \in F' \sqcup M'$.

Suppose $Q$ is a quasi-projective cover of $F' \sqcup M' \sqcup \theta$ with an epimorphism
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$\pi : Q \to F' \sqcup M' \sqcup \theta$. Now define $q : F' \sqcup M' \sqcup \theta \to F' \sqcup \theta$ by

$$q(x) = \begin{cases} x, & x \in F' \sqcup \theta; \\ \theta, & x \in M'. \end{cases} \quad (2)$$

Clearly $q$ is a homomorphism. Now consider the following diagram:

$$\begin{array}{ccc}
F' \sqcup \theta & \xrightarrow{\pi} & F' \sqcup M' \\
\downarrow^{1_{F' \sqcup \theta}} & & \downarrow^{q} \\
Q & \xrightarrow{\pi} & F' \sqcup \theta
\end{array}$$

Since $F' \sqcup \theta \simeq F$ is projective, there exists $i : F' \sqcup \theta \to Q$ such that $q \circ \pi \circ i = 1_{F' \sqcup \theta}$. Thus $i$ is a monomorphism and we can regard $F$ as a subact of $Q$. Let $K = \{ x \in Q : q(\pi(x)) = \theta_F \}$ and $K' = K - \{ \theta_Q \}$. Clearly $K' \sqcup i(F' \sqcup \theta)$ is a subact of $Q$. We show that $\pi_1 = \pi|_{K' \sqcup i(F' \sqcup \theta)} : K' \sqcup i(F' \sqcup \theta) \to F' \sqcup M' \sqcup \theta$ is an epimorphism. For this we show that $\pi(i(x)) = x$, for every $x \in F' \sqcup \theta$. Suppose $x \in F' \sqcup \theta$. If $x = \theta$, then clearly $\pi(i(\theta)) = \theta$. Suppose $x \in F'$ and let $z = \pi(i(x))$. Then $q(z) = q(\pi(i(x))) = x$. Thus $q(z) = x \in F'$. By the definition of $q$, $q(z) = z$, i.e., $z = x$. Thus $\pi(i(x)) = x$ for every $x \in F' \sqcup \theta$. Thus $\pi_1$ is an epimorphism and since $\pi$ is coessential, $Q = K' \sqcup i(F' \sqcup \theta) \simeq K' \sqcup F' \sqcup \theta$. Now let $\pi_2 = \pi|_{K' \sqcup \theta} : (K' \sqcup \theta) \simeq K \to (M' \sqcup \theta) \simeq M$. Since $\pi$ is coessential $\pi_2$ is a coessential epimorphism. Since $F$ is a projective $S$-act, there exists $\phi' : F \to K$ such that the diagram

$$\begin{array}{ccc}
F & \xrightarrow{\phi'} & Q \\
\downarrow^{\pi_2} & & \downarrow^{Q} \\
K & \xrightarrow{\pi_2} & M
\end{array}$$

is commutative and $\pi_2 \circ \phi' = \phi$. Thus $\pi_2(\phi'(F)) = \phi(F) = M$ and, since $\pi_2$ is coessential, $\phi'$ is an epimorphism. Now define $q' : F' \sqcup K' \sqcup \theta \to K' \sqcup \theta$ by

$$q'(x) = \begin{cases} x, & x \in K' \sqcup \theta; \\ \theta, & x \in F'. \end{cases} \quad (3)$$
and $q'' : F' \sqcup K' \sqcup \theta \to F' \sqcup \theta$ by
\[
q''(x) = \begin{cases} 
  x, & x \in F' \sqcup \theta; \\
  \theta, & x \in K'.
\end{cases}
\] (4)

Clearly $q'$ and $q''$ are homomorphism. Now consider the following diagram
\[
\begin{array}{ccc}
F' \sqcup K' \sqcup \theta & \xrightarrow{q''} & (F' \sqcup \theta) \simeq F & \xrightarrow{\phi'} & (K' \sqcup \theta) \simeq K \\
\downarrow q' & & \downarrow h & & \downarrow 1_{K' \sqcup \theta} \\
K' \sqcup \theta & & 1_{K' \sqcup \theta} & & \\
\end{array}
\]

Since $F' \sqcup K' \sqcup \theta \simeq Q$ is quasi-projective, there exists $h : F' \sqcup K' \sqcup \theta \to F' \sqcup K' \sqcup \theta$ such that $\phi' \circ q'' \circ h = 1_{K' \sqcup \theta} \circ q'$. If $j : K' \sqcup \theta \to F' \sqcup K' \sqcup \theta$ is the canonical injection, then $q' \circ j = 1_{K' \sqcup \theta}$ and so $\phi' \circ q'' \circ h \circ j = 1_{K' \sqcup \theta}$. Thus $K \simeq K' \sqcup \theta$ is a retract of $F' \sqcup \theta \simeq F$ and so is projective. Hence $K$ is the projective cover of $M$. \hfill \Box

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References


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